Unit–1

Classical Mechanics (Lagrangian and Hamiltonian)

(Q). Write down the disadvantages of Newtonian Mechanics.

→ (i) To find equation of motion of a system using newtonian mechanics we have to find the total force on the system.

From newtonian mechanics, $m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{net}$

as force is a vector quantity ; it is some times difficult to find the vector sum of all the forces.

(ii). if there is pseudo force on the system it is difficult to apply newtonian mechanics.

~~(iii). write~~

H/W. (Q) write down the limitations of newton's laws of motion.

→

(i) Newton's law are applicable only in the inertial frame of reference.

(ii). Newton's law are applicable for point objects.

(iii). Newton's laws are applicable to rigid bodies.

(iv). For objects moving with speed of comparable to that of light, newton's law of motion failed to give accurate result.

(Q) Define lagrangian function of a system.

⟶ lagrangian functions of a system defined as, $L = T - V$, where T = KE of the system
V = PE of the system.

The lagrangian equⁿ of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad [\text{for conservative sys}]$$

$\dot{q} = \frac{dq}{dt}$

In lagrangian mechanics we have to find the total KE and the total PE of the system.

q = generalised co-ordinate

(Q) write down the advantages of lagrangian mechanics over Newtonian mechanics.

⟶ In newtonian mechanics we have to find the net force applied on the system to get the equation of motion of the system.

In Lagrangian mechanics, we have to find the total KE and the total PE of the sys. to get the equⁿ of motion of the system.

As force is a vector quantity and energy is a scalar quantity it is easier to apply lagrangian mechanics to the system where there is more than three or more forces in diffⁿ direction. [For a complex sys.]

(Q) Using Newtonian mechanics find the equⁿ of motion of a particle going through a viscous medium.

⟶ The viscous force on the particle is proportional to its velocity.

⟹ The viscous force on the particle, $F \propto -v$

$\Rightarrow F = -kv$

From newton's equn of motion $m\frac{d^2r}{dt^2} = -Kv$

$\Rightarrow \frac{d^2r}{dt^2} + \frac{K}{m}v = 0$

$\Rightarrow \frac{d^2r}{dt^2} + \frac{K}{m}\frac{dr}{dt} = 0 \longrightarrow (1)$.

from equn $(m_1^2 + \frac{K}{m}m) = 0$

$\Rightarrow m_1(m_1 + \frac{K}{m}) = 0$

$\therefore m_1 = 0, -\frac{K}{m}$

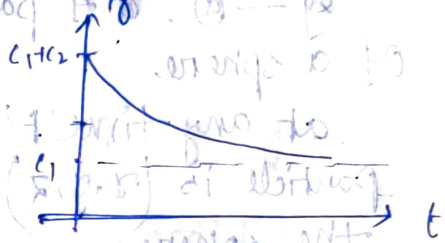
$\therefore r = c_1 e^{0\cdot t} + c_2 e^{-Kt/m}$ $c_1, c_2 \rightarrow$ Arbitrary const

$\Rightarrow r = c_1 + c_2 e^{-Kt/m} \longrightarrow (ii)$.

equn (ii) represent equn of motion of the particle

$t \to 0, r = c_1 + c_2$

$t \to \infty, r = c_1$



(Q) A particle is falling under gravity, the viscous force on the particle is proportional to its velocity. Find the velocity of the particle at any time t. Hence find the velocity of the particle at $t \to \infty$

→ The net force on the particle $\vec{F}_{net} = mg - Kv$

$\Rightarrow m\frac{dv}{dt} = mg - Kv$

$\Rightarrow \frac{dv}{dt} = g - \frac{K}{m}v$

$\Rightarrow \frac{dv}{dt} + \frac{K}{m}v = g \longrightarrow (i)$.

$I.F = e^{\int \frac{K}{m}dt} = e^{\frac{Kt}{m}}$

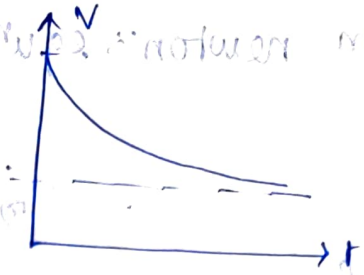
multiplying both side of equn (i) by I.F

$ve^{\frac{Kt}{m}} = \int g e^{\frac{Kt}{m}} dt$

$\Rightarrow ve^{Kt/m} = g\frac{m}{K}e^{Kt/m} + c$

$\Rightarrow v = \frac{mg}{K} + ce^{-Kt/m} \longrightarrow (ii)$.

at $t \to \infty$, $V = \frac{mg}{k}$



(Q) Define constraint of a system.

(ii) Define degree of freedom of a system.

(iii). Define generalised co-ordinate of a system.

→ (i) The restriction imposed on the motion of the particle is known as constrain of the system.

eg — (a). A particle is moving on the the surface of a sphere

at any time 't' the posⁿ of the particle is (x, y, z) w.r to centre of the sphere

$R = \sqrt{x^2+y^2+z^2}$ (R)

The motion of the particle must follow the equⁿ $x^2+y^2+z^2 = R^2$ for any time t. This is the restriction imposed on the motion of the particle.

So, equⁿ of constrain is $x^2+y^2+z^2 = R^2$

(b). The motion of a simple pendulum. Here the constrain is $\boxed{l = \text{const}}$

(c) A particle is moving inside a sphere.

$x^2+y^2+z^2 \leq R^2$

(ii) let, n = total no. of co-ordinate in a system
k = no of constrain in the system.

So, the total no. of independent co-ordinate

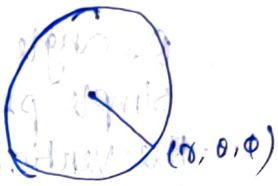
~~$f = n - k$~~ $f = n - k$

This total no. of independent co-ordinate is known as degree of freedom of the system.

This independent co-ordinates are known as generalised co-ordinate.

eg:— A particle is moving on the surface of a sphere.

The constrain in the system dist $r = R$ (fixed).



Independent variables are (θ, ϕ)

Degree of freedom $= 3 - 1 = 2$

θ, ϕ is generalised co-ordinate of the sys.

(Q). Find degree of freedom of the following system

(i) A particle is moving in a 3-D plane

(ii) A particle is moving like a sinusodial wave

(iii) A particle moving on the surface of a sphere

(iv). A cylinder rolling down in an inclined plane.

→ (i) $n = 3, K = 0$

$f = 3 - 0 = 3$

(ii) $n = 2, K = 1$

$f = 1$

(iii). $n = 3, K = 1$

$f = 2$

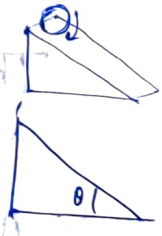
(iv). ~~$n = 3, K = 0$~~

$f = 2$

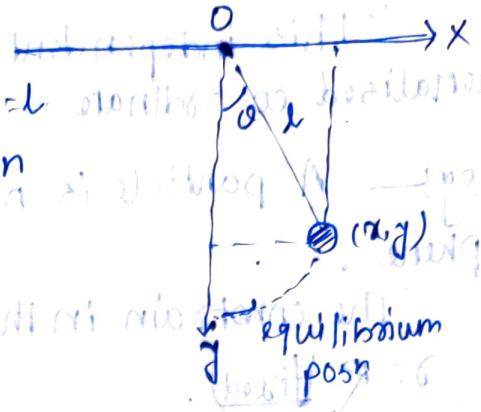
$n = 2, K = 1$

$\theta = \text{const}$

$\therefore f = 1$.



(9). Using Lagrangian mechanics find the equⁿ of motion of simple pendulum.



→ we consider a simple pendulum of length = l at any time 't' the position of the pendulum is (x, y).

θ = angle between the simple pendulum and the vertical line at time 't'.

$x = l\sin\theta$

$y = l\cos\theta$

The KE of the system $= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ $\quad \dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

$\therefore \dot{x} = +l\cos\theta\frac{d\theta}{dt} = +l\cos\theta\cdot\dot{\theta}$

$\dot{y} = -l\sin\theta\cdot\frac{d\theta}{dt} = -l\sin\theta\cdot\dot{\theta}$

$\therefore KE = \frac{1}{2}m\{l^2\cos^2\theta\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\theta}^2\}$

$KE = \frac{1}{2}ml^2\dot{\theta}^2$

Potential energy of the system, $V = -mgy$ ← (Reference x-axis)

$= -mgl\cos\theta$

The Lagrangian funcⁿ of the system, $L = KE - PE$

$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$ ⟶ (1)

here θ = generalised co-ordinate

The Lagrangian equⁿ of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$

$\therefore \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$, $\frac{\partial L}{\partial\theta} = -mgl\sin\theta$.

$\therefore \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$ $\quad \theta = 1°$

$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$

$\Rightarrow ml^2\ddot{\theta} + mg\theta = 0$

$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$ ⟶ (11).

eqn (ii) represent Lagrangian equation of motion for simple pendulum.

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$\therefore \omega^2 = \left(\sqrt{\frac{g}{l}}\right)^2 \quad \& \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

H/W. Define Virtual Displacement.

⟶ Let us consider a system with n no. of generalised co-ordinates $(q_1, q_2, \ldots q_n)$. Now the system is displaced by some amount without taking any time. This type of displacement is known as virtual displacement. $\boxed{\delta t = 0}$ which is impossible.

The work done by the forces for virtual displacement is always zero.

(constraint force) $\vec{F} \cdot \delta\vec{r} = 0$ $\delta\vec{r}$ = virtual displacement

(g) Classify constrains based on time dependency.

⟶ The constraint which are time ~~inp~~ independent is known as Scleronomous constrain

eg:— The length of a ideal simple pendulum.

l = const.

The constrain which are depend on time known as Rehonomous constrain.

eg:— A simple pendulum with length (l) & time

Note:— Based on equation constrain can be classified as ⟶ (i). Holonomic constraint:— The constraints of a system which can be expressed as equation are known as ~~hom~~ Holonomic constraint.

eg:— A particle moving on the surface of a sphere of radius R

The constraint in this case can be represent by

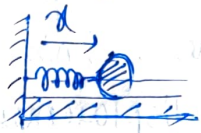
$x^2 + y^2 + z^2 = R^2$.

(ii) Non-holonomic constraint:— The constraint of a system which can not be expressed by an equⁿ is known as non-holonomic constraint.

eg:— A particle moving inside a sphere of Radius R. In this case, the constrain of the system represented by $x^2+y^2+z^2 \leq R^2$

(Q) By using, Lagrangian mechanics find the equⁿ of motion of 1-D simple harmonic motⁿ.

⟶ $L = T - V$ $\quad V = \frac{1}{2}kx^2$



$KE = \frac{1}{2}mv^2$

$\therefore L = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

generalised co-ordinate : x $\quad \dot{x} = \frac{dx}{dt}$

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\Rightarrow \frac{d}{dt}\{m\dot{x}\} - (-kx) = 0$

$\Rightarrow m\frac{d\dot{x}}{dt} + kx = 0$

$\Rightarrow m\ddot{x} + kx = 0$

$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$

$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ This is the equⁿ of SHM in Lagrangian motion

(Q). Show that, in case of rigid body the force of constraint is unable to do any work.
or, ST, W.D by holonomic constrains is zero

⟶ we consider two particles i, K on the rigid body. The distance between this two particles is always constant.

$\vec{r}_i$ = Distance of ith particle from the origin

$\vec{r}_K$ = Distance of the Kth from the origin

So the Distance between ith and Kth particle

$$\boxed{\vec{r}_K - \vec{r}_i = \text{const}} \longrightarrow 0.$$

let, $\vec{F}_{iK}$ is the force on the ith particle due to K^{th} particle.

F_{iK} = Mutual internal force.

The displacement of the ith particle due to F_{iK} force is $d\vec{r}_i$

$$\text{work done} = \vec{F}_{iK} \cdot d\vec{r}_i$$

The workdone in the displacement of the ith particle due to all the particles $\sum_{K=i} \vec{F}_{iK} \cdot d\vec{r}_{iK} = w_i \quad [i \neq K] \longrightarrow (ii)$ [For 1 particle]

The workdone on the all the particles of the sys.

$$W = \sum_{i=1} w_i = \sum_{i=1} \sum_{K=1} \vec{F}_{iK} \cdot d\vec{r}_i \quad (i \neq K) \longrightarrow (iii)$$

[Force by K particle to all particles]

As, $\vec{F}_{iK}$ is mutual interaction force, $\vec{F}_{iK} = -\vec{F}_{Ki} \longrightarrow (iv)$

F_{Ki} = force on the K^{th} particle due to ith particle.

From equ^n (iii) $\Rightarrow$ $W = \sum_{K=1} w_K = \sum_{K=1} \sum_{i=1} \vec{F}_{Ki} \cdot d\vec{r}_K \quad (K \neq i)$

$\longrightarrow (v)$.

(iii) + (v) $\Rightarrow$ $\sum_{i=1} w_i + \sum_{K=1} w_K$

$$\Rightarrow 2W = \sum_{i,K=1} \vec{F}_{iK} \cdot d\vec{r}_i + \vec{F}_{Ki} \cdot d\vec{r}_K \quad (i \neq K)$$

$$= \sum_{i,K=1} \vec{F}_{iK} \cdot d\vec{r}_i - \vec{F}_{iK} \cdot d\vec{r}_K$$

$$= \sum_{i,K=1} \vec{F}_{iK} \cdot (d\vec{r}_i - d\vec{r}_K)$$

$$\therefore W = \frac{1}{2} \sum_{i,K=1} \vec{F}_{iK} (d\vec{r}_i - d\vec{r}_K) \longrightarrow (vi).$$

$\vec{F}_{12} = -\vec{F}_{21}$

$r_k - r_i = \text{const}$

$\Rightarrow dr_k - dr_i = 0$

$\therefore$ from (vi) $\Rightarrow W = 0$

so, the work done by constraint force in a rigid body is always zero.

(Q) Find Lagrangian equn of motion of a particle in planetary motion.

$\longrightarrow$ We know that planetary motion is confined to a plane. A particle of mass m is in planetary motion. KE of the particle $= \frac{1}{2}mv^2$ $\quad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= \frac{1}{2}m\vec{v}\cdot\vec{v}$

$= \frac{1}{2}m\{\dot{r}^2 + r^2\dot{\theta}^2\}$

potential energy of particle, $V = V(r)$

Lagrangian of the sys: ~~for generalised co-ordinate~~

$L = T - V$

$= \frac{1}{2}m\{\dot{r}^2 + r^2\dot{\theta}^2\} - V(r)$

Here r, θ generalised co-ordinate.

Lagrangian equn of motion for generalised co-ordinate r; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ $\quad \left| \frac{\partial L}{\partial \dot{r}} = m\dot{r} \right.$ $\quad \left. \frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 \right.$

$\Rightarrow \frac{d}{dt}(m\dot{r}) + \frac{\partial V}{\partial r} - mr\dot{\theta}^2 = 0$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$

$\Rightarrow \boxed{m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r}}$

for generalised co-ordinate, θ,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ $\quad \left| \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l \right.$ $\quad \left. \frac{\partial L}{\partial \theta} = 0 \right.$ = Any

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \hookrightarrow \frac{dl}{dt} = 0$, $l = \text{const}$ Angular momentum

$\Rightarrow$ ~~$mr^2\ddot{\theta} = 0$~~ $mr^2\ddot{\theta} + mi\,2r\dot{r} = 0$

$\Rightarrow mr^2\ddot{\theta} + 2mr\dot{\theta}\dot{r} = 0$

(Q). By using Lagrangian mechanics find the equn of motion of a projectile. Hence ST it's path will be parabolic.

$\longrightarrow$ at any time t the projectile at (x, y)

$\therefore$ KE of the particle $= \frac{1}{2}mv^2$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

P.E of the particle $= mgh$

$\therefore L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgh$

$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{\partial L}{\partial y} = -mg$

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$

$\Rightarrow \frac{d}{dt}m\dot{y} + mg = 0$

$\Rightarrow m\ddot{y} = -mg \Rightarrow m\frac{d^2y}{dt^2} = -mg \Rightarrow \ddot{y} = -g \longrightarrow (1)$

Again, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\Rightarrow \frac{d}{dt}(m\dot{x}) - 0 = 0$

$\Rightarrow m\dot{x} = 0 \Rightarrow \ddot{x} = 0 \longrightarrow (II)$

From equn (II) $\Rightarrow \frac{d^2x}{dt^2} = 0$

$\frac{dx}{dt}$ = velo x axis

integrating both side, $\frac{dx}{dt} = c_1$

at $t = 0$ x component of velocity, $v\cos\theta$

$c_1 = v\cos\theta \quad \therefore \frac{dx}{dt} = v\cos\theta \longrightarrow (III)$

again integrating, $x = v\cos\theta\, t + c_2$

at $t = 0$, $x = 0$

$c_2 = 0$

$x = v\cos\theta \cdot t \longrightarrow (IV)$

$\Rightarrow t = \frac{x}{v\cos\theta}$

from equⁿ (i) ⇒ $\frac{d^2y}{dt^2} = -g$

integrating both side, $\frac{dy}{dt} = -gt + c_3$

at $t = 0$, $\frac{dy}{dt} = v\sin\theta$

$\therefore c_3 = v\sin\theta$

$\therefore \frac{dy}{dt} = -gt + v\sin\theta$

again integrating, $y = -gt^2/2 + vt\sin\theta + c_4$

at $t = 0$, $y = 0$, $\therefore c_4 = 0$

$\therefore y = -g\frac{t^2}{2} + v\sin\theta t$

from (iv) ⇒ $y = -g\cdot\left(\frac{x}{2v\cos\theta}\right)^2 + v\sin\theta\cdot\frac{x}{v\cos\theta}$

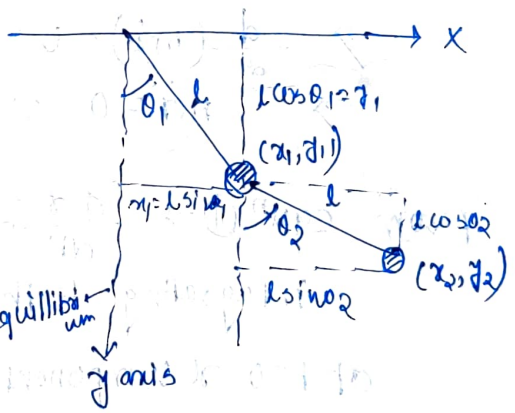
$y = -g\frac{x^2}{2v^2\cos^2\theta} + x\tan\theta$ ⟶ (v) equⁿ of parabola.

equⁿ (v) represent equation of a parabola.

(Q) Find Lagragian equⁿ of motion for a double pendulum system with identical pendulum length.

⟶ we consider a double pendulum system consist of two identical simple pendulum.

The point of suspension of the 2nd pendulum is the bob of 1st pendulum.



The length of the pendulum = l

mass of the bob = m

At any time 't' the position of 1st bob (x_1, y_1) and posⁿ of 2nd bob (x_2, y_2)

From fig, $x_1 = l\sin\theta_1$, $x_2 = l\sin\theta_1 + l\sin\theta_2$

$y_1 = l\cos\theta_1$, $y_2 = l\cos\theta_1 + l\cos\theta_2$

The KE of the system $= \frac{1}{2}m(\dot{x}_1\hat{i} + \dot{y}_1\hat{j}) + \frac{1}{2}m(\dot{x}_2\hat{i} + \dot{y}_2\hat{j})$

$KE = \frac{1}{2}m\left[l^2\cos^2\theta_1\dot{\theta}_1^2 + l^2\sin^2\theta_1\dot{\theta}_1^2\right] + \frac{1}{2}m\left[(l\cos\theta_1\dot{\theta}_1 + l\cos\theta_2\dot{\theta}_2)^2 + (l\sin\theta_1\dot{\theta}_1 + l\sin\theta_2\dot{\theta}_2)^2\right]$

$\dot{x}_1 = l\cos\theta_1\dot{\theta}_1$
$\dot{y}_1 = -l\sin\theta_1\dot{\theta}_1$
$\dot{x}_2 = (l\cos\theta_1 + l\cos\theta_2)\dot{\theta}_2$
$\dot{y}_2 = -(l\sin\theta_1 + l\sin\theta_2)\dot{\theta}_2$

$KE = \frac{1}{2}m\left[l^2\dot{\theta}_1^2\right] + \frac{1}{2}m\left[l_1^2\cos^2\theta_1\dot{\theta}_1^2 + l^2\cos^2\theta_2\dot{\theta}_2^2 + 2l^2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + l^2\sin^2\theta_1\dot{\theta}_1^2 + l^2\cos^2\theta_2\dot{\theta}_2^2 + 2l^2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2\right]$

$= \frac{m}{2}l^2\dot{\theta}_1^2 + \frac{m}{2}\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right]$

$\Rightarrow \frac{ml^2\dot{\theta}_1^2}{2} + \frac{m}{2}\left[l_1^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$

$PE = -mgy_1 - mgy_2$
$= -mgl\cos\theta_1 - mg(l\cos\theta_1 + l\cos\theta_2)$
$PE = -2mgl\cos\theta_1 - mgl\cos\theta_2$

$L = KE - PE$
$= \frac{ml^2\dot{\theta}_1^2}{2} + \frac{m}{2}\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + 2mgl\cos\theta_1 + mgl\cos\theta_2$

Generalised co-ordinates are θ_1 and θ_2

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0$

$\Rightarrow \frac{d}{dt}\left\{2l^2\dot{\theta}_1 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_2\right\}m + 2mgl\sin\theta_1 - 2l^2\sin\theta_2\cos\theta_1\dot{\theta}_1\dot{\theta}_2 + 2l^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$

$\Rightarrow m(2l^2\ddot{\theta}_1) + 2l^2\cos(\theta_1 - \theta_2)\ddot{\theta}_2 \cdot m + 2mgl\sin\theta_1 - 2l^2\sin\theta_2\cos\theta_1\dot{\theta}_1\dot{\theta}_2 \cdot m + 2l^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$

For co-ordinate θ_2; $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0$

$\Rightarrow \frac{d}{dt}\left\{ml^2\dot{\theta}_2 + 2l^2\dot{\theta}_1\cos(\theta_1 - \theta_2)\right\}m + mgl\sin\theta_2 + 2l^2\cos\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 - 2l^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$

$\Rightarrow ml^2\ddot{\theta}_2 + 2l^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) \cdot m + mgl\sin\theta_2 + 2l^2m\cos\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 - 2ml^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$

For generalised co-ordinate θ_1,

$$\frac{d}{dt}\left\{2mL^2\dot{\theta}_1 + 2mL^2\cos(\theta_1-\theta_2)\dot{\theta}_2\right\} + 2mgL\sin\theta_1 - 2L^2\sin\theta_2\cos\theta_1\dot{\theta}_1\dot{\theta}_2 + 2L^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$$

$$\Rightarrow 2mL^2\ddot{\theta}_1 + 2mL^2\ddot{\theta}\cos(\theta_1-\theta_2) + 2mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + u = 0$$

$$\Rightarrow \ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \sin(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2 = 0 \longrightarrow (1)$$

$$\Rightarrow 2mL^2\ddot{\theta}_1 + 2mL^2\ddot{\theta}\cos(\theta_1-\theta_2) + 2mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + 2mgL\sin\theta_1 - 2L^2\sin\theta_2\cos\theta_1\dot{\theta}_1\dot{\theta}_2 + 2L^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0 \longrightarrow (1).$$

For generalised co-ordinate $\dot{\theta}_2$,

$$\frac{d}{dt}\left(mL^2\dot{\theta}_2 + 2mL^2\dot{\theta}_1\cos(\theta_1-\theta_2)\right) + mgL\sin\theta_2 + 2L^2\cos\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 - 2L^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$$

$$\Rightarrow mL^2\ddot{\theta}_2 + 2mL^2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + 2mL^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + mgL\sin\theta_2 + 2L^2\cos\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 - 2L^2\sin\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 = 0$$

16-05

(Q). Define generalised velocity, generalise momentum, generalised force

(ii). Define cylic co-ordinate. Write down its importance.

⟶ (i). We consider a system with Lagrangian funcⁿ

$$L = \frac{m\dot{q}^2}{2} - V(q) \longrightarrow (1).$$

Here q = generalised co-ordinate
m = mass of the particle
For conservative for. $V = V(q)$

The quantity $\boxed{\dot{q} = \frac{dq}{dt}}$ is known as generalised velocity.

The generalised momentum of the system is given by $\boxed{p = m\dot{q}}$

The generalised force of the system is given by,

$$\boxed{F = \frac{d}{dt}(m\dot{q})}$$

from equation (1) $\frac{\partial L}{\partial \dot{q}} = m\dot{q}$ = Generalised momentum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}(m\dot{q}) = \text{Force}$$

$$\frac{\partial L}{\partial q} = \text{generalised force.}$$

(ii). Let, q is one of the generalised co-ordinate of a system with Lagragian L. If $\frac{\partial L}{\partial q} = 0$ then q is called cylic co-ordinate.

Imp:- If a generalised co-ordinate is cylic then the generalised momentum corresponding to these generalised co-ordinate will be conserved.

$$\frac{\partial L}{\partial \dot{q}} = \text{const}$$

(Q). The Lagrangian of a system is given by L⊖,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + \frac{K}{r^2} \quad : K = \text{const}$$

(i). Find the generalised momentum corresponding to each generalised co-ordinate.

(ii). Find the cylic co-ordinate in this system.

⟶ generalised momentum will be

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

again $\frac{\partial L}{\partial r} = \frac{m}{2}\dot{\theta}^2 - \frac{2K}{r^3}$

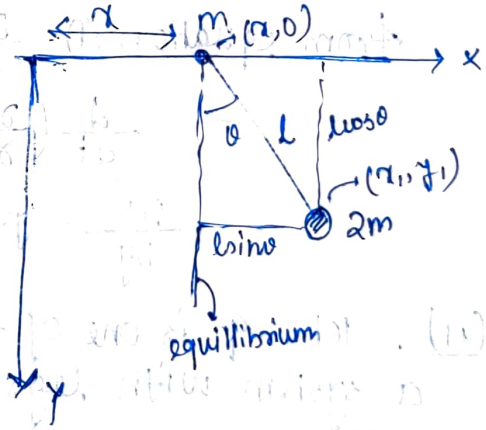
$$\frac{\partial L}{\partial \theta} = \cancel{2}\sin\theta\cos\theta\,\frac{m}{\cancel{2}}\,r^2\dot{\phi}^2$$

$$= mr^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$\frac{\partial L}{\partial \phi} = 0$ ~~This th~~ ∴ ϕ is ~~gener~~ cylic co-ordinate

(Q). A simple pendulum has length l and mass of the bob is $2m$. The point of suspension of the simple pendulum has mass m. The point of suspension is moving in the horizantal direction. Find the Lagragian eqn of motion of the sys.

⟶ let at time t the posn of the point suspension w.r.t origin is $(x,0)$ and the position of bob is (x_1, y_1).



From fig, $x_1 = x + l\sin\theta$

$y_1 = l\cos\theta$

The KE of the sys is given by $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}2m(\dot{x}_1^2 + \dot{y}_1^2)$

$\dot{x}_1 = \dot{x} + l\cos\theta\,\dot{\theta}$

$\dot{y}_1 = -l\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2}m\dot{x}^2 + m\{\dot{x}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{x}l\cos\theta\,\dot{\theta} + l^2\sin^2\theta\,\dot{\theta}^2\}$$

$$= \frac{1}{2}m\dot{x}^2 + m\{\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\cos\theta\,\dot{\theta}\}$$

$$= \frac{3}{2}m\dot{x}^2 + ml^2\dot{\theta}^2 + 2m\dot{x}l\cos\theta\,\dot{\theta}$$

PE of the sys, $V = 0 - mgy_1$

$V = -2mg\cos\theta$

$\therefore L = \frac{3}{2}m\dot{x}^2 + ml^2\dot{\theta}^2 + 2ml\dot{\theta}\dot{x}\cos\theta + mgl\cos\theta$

$\therefore$ Lagrangian equn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

generalised co-ordinate : x and θ

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ $\quad \frac{\partial L}{\partial \dot{x}} = 3m\dot{x} + 2ml\dot{\theta}\cos\theta$

$\Rightarrow \frac{d}{dt}(3m\dot{x} + 2ml\dot{\theta}\cos\theta) = 0$ $\quad \frac{\partial L}{\partial x} = 0$

$\Rightarrow 3m\ddot{x} + 2ml\ddot{\theta}\cos\theta - 2ml\dot{\theta}^2\sin\theta = 0 \longrightarrow (1)$

For θ, $\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \theta} = 0$ $\quad \frac{\partial L}{\partial \dot{\theta}} = 2ml^2\dot{\theta} + 2ml\dot{x}\cos\theta$

$\frac{\partial L}{\partial \theta} = -2mgl\sin\theta - 2ml\dot{\theta}\dot{x}\sin\theta$

$\Rightarrow \frac{d}{dt}(2ml^2\dot{\theta} + 2ml\dot{x}\cos\theta) + 2mgl\sin\theta = 0$

$\Rightarrow 2ml\{\dot{x}(-\sin\theta)\dot{\theta} + \cos\theta\cdot\ddot{x}\} + 2mgl\sin\theta + 2ml\dot{\theta}\dot{x}\sin\theta$

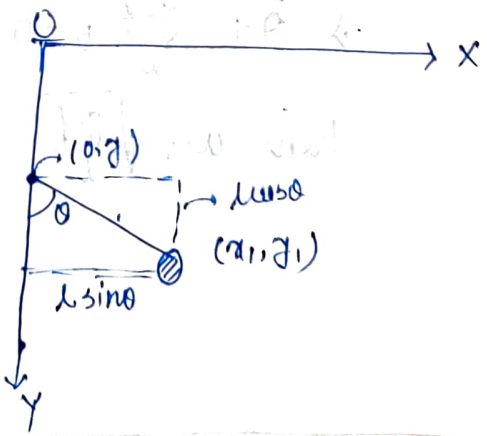
$\Rightarrow 2ml\cos\theta\ddot{x} - 2ml\dot{x}\sin\theta\dot{\theta} + 2mgl\sin\theta = 0$

$\Rightarrow 2ml^2\ddot{\theta} - 2ml\dot{x}\sin\theta\dot{\theta} + 2ml\ddot{x}\cos\theta + 2mgl\sin\theta + 2ml\dot{\theta}\ddot{x}\sin\theta = 0$

$\Rightarrow 2ml^2\ddot{\theta} + 2ml\ddot{x}\cos\theta + 2mgl\sin\theta = 0$

$\Rightarrow l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \longrightarrow (11)$.

(Q) A simple pendulum has length l and mass of the bob is small m. If the point of suspension of the simple pendulum moving vertically downward with accelaration 'f', Find the Lagrangian equn of motion of the system and time period of the simple pendulum.



$\longrightarrow$ Let the posn of the point of suspension is $(0, y)$ and posn of the bob is (x_1, y_1) at any time 't'.

From the fig, $x_1 = l\sin\theta$

$y_1 = y + l\cos\theta$

$s = ut + \frac{1}{2}at^2$ $\quad f \to$ uniform

$\Rightarrow y = 0 + \frac{1}{2}ft^2$ $\quad \dot{x}_1 = l\cos\theta\,\dot{\theta}$

$\dot{y}_1 = ft - l\sin\theta\,\dot{\theta}$

$\therefore \quad y_1 = \frac{1}{2}ft^2 + l\cos\theta$

KE of the system, $= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$

$= \frac{1}{2}m\left[l^2\cos^2\theta\,\dot{\theta}^2 + f^2t^2 + l^2\sin^2\theta\,\dot{\theta}^2 - 2ftl\sin\theta\,\dot{\theta}\right]$

$KE, T = \frac{m}{2}\left[l^2\dot{\theta}^2 + f^2t^2 - 2ftl\sin\theta\,\dot{\theta}\right]$

PE of the sys, $PE, V = -mgy_1$

$= -mg\left(\frac{1}{2}ft^2 + l\cos\theta\right)$

$L = \frac{m}{2}\left[l^2\dot{\theta}^2 + f^2t^2 - 2ftl\sin\theta\,\dot{\theta}\right] + mg\left\{\frac{1}{2}ft^2 + l\cos\theta\right\}$

$\therefore$ Lagrangian equn for generalised co-ordinate θ

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - ftl\sin\theta \cdot \frac{m}{2} = mftl$

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta - \frac{m}{2} \times 2ftl\cos\theta$

$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta} - mftl\sin\theta) + mgl\sin\theta + (mftl\cos\theta\,\dot{\theta}) = 0$

$\Rightarrow$ ~~$ml^2\ddot{\theta} + mftl\cos\theta\,\dot{\theta} + mgl\sin\theta + mftl\cos\theta\,\dot{\theta} = 0$~~

$\Rightarrow ml^2\ddot{\theta} - mftl\sin\theta - mftl\cos\theta\,\dot{\theta} + mgl\sin\theta + mftl\cos\theta\,\dot{\theta} = 0$ (2)

$\Rightarrow ml^2\ddot{\theta} - mftl\sin\theta + mgl\sin\theta = 0$

$\Rightarrow \ddot{\theta} - \frac{ftl\sin\theta}{l} + \frac{gl\sin\theta}{l} = 0$

$\Rightarrow \ddot{\theta} + \frac{g-f}{l}\sin\theta = 0$ $\quad \theta$ is small $\therefore \sin\theta \simeq \theta$

$\Rightarrow \ddot{\theta} + \frac{g-f}{l}\theta = 0$

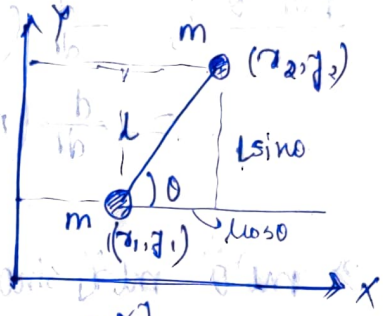
here $\omega = \sqrt{\frac{g-f}{l}}$ $\quad \therefore$ Time period $= \frac{2\pi}{\omega}$

$T = 2\pi\sqrt{\frac{l}{g-f}}$

(d). same ques as previous, just vertically upward.

⟹ Ans:- $\ddot{\theta} + \frac{g+f}{l}\theta = 0$

(e) H/W A rigid dumbbell of length l consist of two equal mass m connected by massless rod. If the dumbbell rotate about its C.O.M in x-y plane. Find the Lagrangian equation of motion of the system.



⟹ $x_2 = x_1 + l\cos\theta$

$y_2 = y_1 + l\sin\theta$

The total KE of the system

$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$

$= \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}\left[(\dot{x}_1 - l\sin\theta\dot{\theta})^2 + (\dot{y}_1 + l\cos\theta\dot{\theta})^2\right]$

$= \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}\left[\dot{x}_1^2 + l^2\sin^2\theta\dot{\theta}^2 - 2\dot{x}_1 l\sin\theta\dot{\theta} + \dot{y}_1^2 + l^2\cos^2\theta\dot{\theta}^2 + 2\dot{y}_1 l\cos\theta\dot{\theta}\right]$

$= m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}\left[l^2\dot{\theta}^2 - 2\dot{x}_1 l\sin\theta\dot{\theta} + 2\dot{y}_1 l\cos\theta\dot{\theta}\right]$

The PE of the system, $PE = mgy_1 + mgy_2$

$= mgy_1 + mgy_1 + mgl\sin\theta$

$= 2mgy_1 + mgl\sin\theta$

$L = m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}\left[l^2\dot{\theta}^2 - 2\dot{x}_1 l\sin\theta\dot{\theta} + 2\dot{y}_1 l\cos\theta\dot{\theta}\right] + 2mgy_1 - mgl\sin\theta$

Generalised co-ordinate are 3,

$\therefore$ equation for x_1, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$

$\Rightarrow \frac{d}{dt}(2m\dot{x}_1 - ml\dot{\theta}\sin\theta) - 0 = 0$

$\Rightarrow 2m\ddot{x}_1 - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta = 0$

$\Rightarrow 2\ddot{x}_1 - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta = 0$ ——— (1)

For y_1 : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$

$\Rightarrow \frac{d}{dt}(2m\dot{y}_1 + ml\dot{\theta}\cos\theta) + 2mg = 0$

$\Rightarrow 2m\ddot{y}_1 + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + 2mg$

$\Rightarrow 2\ddot{y}_1 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta + 2g = 0$ ——— (ii).

For θ : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$= \frac{d}{dt}\left(ml^2\dot{\theta} - ml\dot{x}_1\sin\theta + ml\dot{y}_1\cos\theta\right) + mgl\cos\theta$

$+ ml\dot{x}_1\dot{\theta}\cos\theta + ml\dot{y}_1\dot{\theta}\sin\theta = 0$

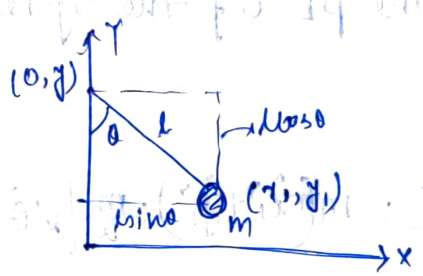
$\Rightarrow ml^2\ddot{\theta} - ml\ddot{x}_1\sin\theta - ml\dot{x}_1\cos\theta\dot{\theta} + ml\dot{y}_1\cos\theta - ml\dot{y}_1\sin\theta\dot{\theta}$

$+ mgl\cos\theta + ml\dot{x}_1\dot{\theta}\cos\theta + ml\dot{y}_1\dot{\theta}\sin\theta = 0$

$\Rightarrow ml^2\ddot{\theta} - ml\ddot{x}_1\sin\theta + ml\ddot{y}_1\cos\theta + mgl\cos\theta = 0$

$\Rightarrow l^2\ddot{\theta} - l\ddot{x}_1\sin\theta + l\ddot{y}_1\cos\theta + gl\cos\theta = 0$

(g). Vertically opposite



$x_1 = l\sin\theta$

$y_1 = y - l\cos\theta$

$y = ut + \frac{1}{2}at^2$

$\Rightarrow y = \frac{1}{2}ft^2$

$\therefore y_1 = \frac{1}{2}ft^2 - l\cos\theta$

$\therefore K.E, T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right)$

$= \frac{m}{2}\left\{l^2\dot{\theta}^2 + 2ft\,l\sin\theta\,\dot{\theta} + f^2t^2\right\}$

$P.E, V = mgy_1$

$= mg\left(\frac{1}{2}ft^2 - l\cos\theta\right)$

$L = \frac{m}{2}\left[l^2\dot{\theta}^2 + 2ftl\sin\theta\,\dot{\theta} + f^2t^2\right] - mg\left(\frac{1}{2}ft^2\right) + mgl\cos\theta$

$= 0$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + 2flt\sin\theta$

$\frac{\partial L}{\partial \theta} = mflt\cos\theta\dot{\theta} - mgl\sin\theta$

$\therefore \frac{d}{dt}(ml^2\dot{\theta} + 2flt\sin\theta) - 2flt\cos\theta\dot{\theta} + mgl\sin\theta = 0$

$\Rightarrow ml^2\ddot{\theta} + 2flt\cos\dot{\theta} + 2flt\sin\theta - 2flt\cos\theta\dot{\theta} + mgl\sin\theta = 0$

$\Rightarrow \ddot{\theta} + \frac{lt\sin\theta}{l} + \frac{g\sin\theta}{l} = 0$

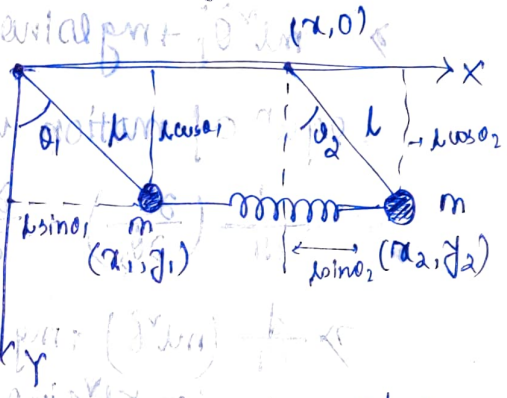
$\Rightarrow \ddot{\theta} + \left(\frac{g+f}{l}\right)\sin\theta = 0$

20-05

(9). Find Lagrangian equation of motion for a parallel pendulum system consist of two identical pendulum (simple) and a ideal spring.

→ We consider a parallel pendulum system consist of two identical simple pendulum of length l and mass of bob is m.

The spring const = K



The distance between the points of suspension of these two pendulum is x.

x = const

From fig. $x_1 = l\sin\theta_1$; $x_2 = x + l\sin\theta_2$

$y_1 = l\cos\theta_1$; $y_2 = l\cos\theta_2$

The K.E of the system = $\frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}(\dot{x}_2^2 + \dot{y}_2^2)$

$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{m}{2}\{l^2\dot{\theta}_2^2\}$

$= \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2$

The PE of the system = $-mgy_1 - mgy_2 + \frac{1}{2}K\{x - (x_2 - x_1)\}$

$= -mgl\cos\theta_1 - mgl\cos\theta_2 + \frac{K}{2}\{x - x - l\sin\theta_2 + l\sin\theta_1\}$

$= -mgl\cos\theta_1 - mgl\cos\theta_2 + \frac{K}{2}\{l^2\sin^2\theta_1 + l^2\sin^2\theta_2 - 2l^2\sin\theta_1\sin\theta_2\}$

$\therefore L = T - V$

$L = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{m}{2}l^2\dot{\theta}_2^2 + mgl\cos\theta_1 + mgl\cos\theta_2 - \frac{K}{2}\{l^2\sin^2\theta_1 + l^2\sin^2\theta_2 - 2l^2\sin\theta_1\sin\theta_2\}$

Lag. eqn of motion w.r.to generalised co-ordinate θ_1

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$

$\frac{\partial L}{\partial \dot{\theta}_1} = ml^2\dot{\theta}_1$

$\frac{\partial L}{\partial \theta_1} = -mgl\sin\theta_1 - \frac{Kl^2}{2}2\sin\theta_1\cos\theta_1 + \frac{2K}{2}l^2\sin\theta_2\cos\theta_1\dot{\theta}_1$

$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta}_1) + mgl\sin\theta_1 + Kl^2\sin\theta_1\cos\theta_1 - Kl^2\sin\theta_2\cos\theta_1 = 0$

$\Rightarrow ml^2\ddot{\theta}_1 + mgl\sin\theta_1 + Kl^2\sin\theta_1\cos\theta_1 - Kl^2\sin\theta_2\cos\theta_1 = 0$

eqn of motion w.r.to co-ordinate θ_2

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$

$\frac{\partial L}{\partial \dot{\theta}_2} = ml^2\dot{\theta}_2$

$\frac{\partial L}{\partial \theta_2} = -mgl\sin\theta_2 - \frac{Kl^2}{2}2\sin\theta_2\cos\theta_2 + Kl^2\sin\theta_1\cos\theta_2$

$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta_2 + Kl^2\sin\theta_2\cos\theta_2 - Kl^2\sin\theta_1\cos\theta_2 = 0$

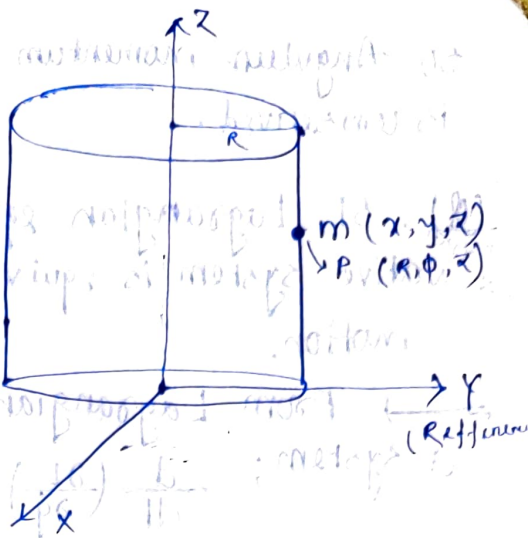
$\Rightarrow ml^2\ddot{\theta}_2 + mgl\sin\theta_2 + Kl^2\sin\theta_2\cos\theta_2 - Kl^2\sin\theta_1\cos\theta_2 = 0$

(Q) A particle of mass m is constraint to move on the curve surface of a cylinder of radius R. If the axis of the cylinder is along z axis and base in x-y plane, find the Lagrangian equation of motion of the sys. Hence ST angular momentum about z axis is const of motion (conserved)

→ At time t the particle at point $P(x,y,z)$.

m = particle's mass

As the particle moving on the curved surface of the cylinder. The posn of the particle at time t can be written in terms of cylindrical co-ordinate (R, ϕ, z)



From curvilinear co-ordinate system, $x = R\cos\phi$ $z = z$

$y = R\sin\phi$

The KE of the particle, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$T = \frac{m}{2}\left\{R^2\dot{\phi}^2(\cos^2\phi + \sin^2\phi) + \dot{z}^2\right\}$$

$$T = \frac{m}{2}\left\{R^2\dot{\phi}^2 + \dot{z}^2\right\}$$

PE of the sys, $V = mgz$

$\therefore L = \frac{m}{2}(R^2\dot{\phi}^2 + \dot{z}^2) - mgz$

Lagrangian equn of motion w.r.t generalised co-ordinate ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi} \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{dt}(mR^2\dot{\phi}) = 0$$

$$\Rightarrow \ddot{\phi} = 0$$

For z $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \frac{\partial L}{\partial z} = -mg$

$\therefore \ddot{z} = -g$

Here ϕ is cylic co-ordinate as $\frac{\partial L}{\partial \phi} = 0$

$\therefore \frac{d}{dt}(mR^2\dot{\phi}) = 0 \Rightarrow mR^2\dot{\phi} = \text{const}$

so Angular momentum of the particle about z axis is conserved..

(Q) ST, Lagrangian eqn of motion for a conservative system is equivalent to Newton's eqn of motion.

⟹ From Lagrangian equation of motion for a system; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$; L = Lagrangian func of sys, q = generalised co-ordinate.

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$ ⟶ (1).

$\frac{\partial L}{\partial \dot{q}}$ = Generalised momentum

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$ = " Force

$\frac{\partial L}{\partial q}$ = generalised force.

The LHS of eqn (1) and RHS of eqn (1) represent Generalised force on the particle. This is similar to newton's eqn of motion

(Q) A particle of mass m moves in 1-D such that it has the Lagrangian $L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$ where V(x) = func of x and $\dot{x}$ = generalised velocity.

Find the Lagrangian eqn of motion of the system.

⟶ Lagrangian eqn of motion w.r.to generalised co-ordinate x

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)\right\} - m\dot{x}^2\frac{\partial V}{\partial x} + 2V\frac{\partial V}{\partial x} = 0$$

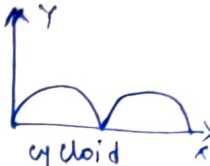
$$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\dot{x}V(x) + 2m\dot{x}\frac{d}{dt}(V(x)) - m\dot{x}^2\frac{\partial V}{\partial x} + 2V\frac{\partial V}{\partial x} = 0$$

$$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\dot{x}V(x) + 2m\dot{x}^2\frac{\partial V}{\partial x} - m\dot{x}^2\frac{\partial V}{\partial x} + 2V\frac{\partial V}{\partial x} = 0$$

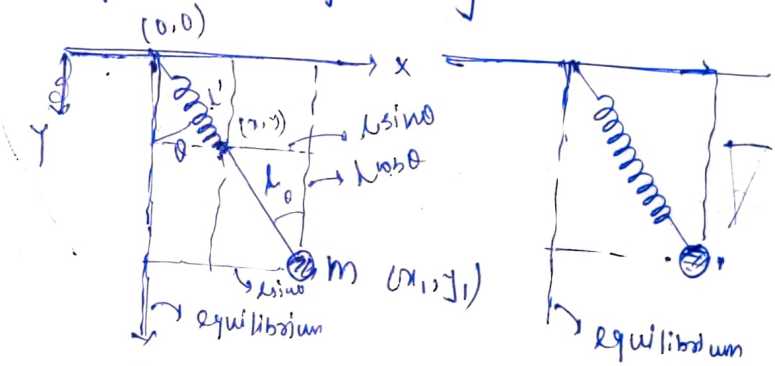
$$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\dot{x}V(x) + m\dot{x}^2\frac{\partial V}{\partial x} + 2V\frac{\partial V}{\partial x} = 0 \quad \longrightarrow (1).$$

$$\frac{d}{dt}(V(x)) = \frac{dV}{dt}\frac{dx}{dt} = \dot{x}\frac{dV}{dx}$$

Hw (g) A bead of mass m slides on a wire described by equation $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$ where, $0 < \theta \leq 2\pi$. Find the Lagrangian equn of motion of the system.



(g) A simple pendulum of length l is connected by a ideal spring of spring const K with the point of suspension. Find the Lagrangian of the system and Lagrangian equation of motion of the sys.



$$x_1 = x + l\sin\theta$$
$$y_1 = y + l\cos\theta$$

$$KE = \frac{m}{2}\left\{\dot{x}_1^2 + \dot{y}_1^2\right\}$$

$$= \frac{m}{2}\left\{(\dot{x} + l\cos\theta\,\dot{\theta})^2 + (\dot{y} - l\sin\theta\,\dot{\theta})^2\right\}$$

$$\dot{x}_1 = \dot{x} + l\cos\theta\,\dot{\theta}$$
$$\dot{y}_1 = \dot{y} - l\sin\theta\,\dot{\theta}$$

$$= \frac{m}{2}\left\{\dot{x}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{x}l\cos\theta\,\dot{\theta} + \dot{y}^2 + l^2\sin^2\theta\,\dot{\theta}^2 - 2l\dot{y}\sin\theta\,\dot{\theta}\right\}$$

$$= \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\cos\theta\,\dot{\theta} - 2l\dot{y}\sin\theta\,\dot{\theta}\right\}$$

P.E, $V = -mgy_1 + \frac{1}{2}K\left[x^2 + y^2\right] + \frac{1}{2}K(l_0 - l')^2$

$= -mg(y + l\cos\theta) + \frac{1}{2}K\left(l_0 - \sqrt{x^2+y^2}\right)^2$

l_0 = initial length of spring
l' = at t length of spring $= \sqrt{x^2+y^2}$

(Q) Define Hamiltonian function of a system.

(ii) write down the relⁿ betⁿ lagrangian funcⁿ and Hamiltonian function of a sys.

⟶ (i). The hamiltonian funcⁿ of a system define as sum of KE and PE of a system.

Hamiltonian funcⁿ, $H = T + V$; T = KE of sys, V = PE " "

(ii). The relation between Lagrangian of a system (L) and Hamiltonian of a sys (H)

$\dot{q}_i$ = generalised velocity

$$H = \sum_{i=1} \dot{q}_i p_i - L$$

p_i = " momentum corresponding to the gen. co-ordinate.

Note :- Lagrangian of a sys $L = L(q_i, \dot{q}_i, t)$

Hamiltonian of a sys, $H = H(q_i, p_i, t)$

(Q) .. Derive Hamiltonian function from Lagrangian eqⁿ of motion

⟶ we consider a conservative system where $L = L(q_i, \dot{q}_i)$ ⟶ (i).

where L = Lagrangian funcⁿ of a sys

q_i = generalised co-ordinate.

$\dot{q}_i$ = " velocity

$f = f(x, y)$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$$\therefore dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \longrightarrow (ii).$$

$$\text{so, } \frac{dL}{dt} = \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \longrightarrow (3).$$

Lagrangian eqⁿ of for a conservative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\text{or } \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \longrightarrow (4).$$

using (4) in eqⁿ (3) ⇒

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$H = p_i \dot{q}_i - L$

$p_i = m\dot{q}_i$

$\therefore H = m\dot{q}\dot{q} - L$

$= 2T - (T - V)$

$\boxed{H = T + V}$

$$\frac{\partial L}{\partial t} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \longrightarrow (5).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right\} = 0 \qquad p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{generalised momentum}$$

$$\Rightarrow \frac{d}{dt}\left\{p_i \dot{q}_i - L\right\}$$

$p_i \dot{q}_i - L = \text{const}$; which is known as Hamiltonian of a system

$$\boxed{H = p_i \dot{q}_i - L}$$

(Q) Derive Hamiltonian canonical equn from Lagrangian of a system. OR,

Sp, Hamiltonian equn of motion is given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$\longrightarrow$ we know that Hamiltonian of $H = H(q_i, p_i, t)$

$$\therefore dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt \longrightarrow (1).$$

If L is the Lagrangian of the system then,

$$H = \dot{q}_i p_i - L \longrightarrow (11).$$

differentiating both side of equn (11).

$$\therefore dH = p_i dq_i + \dot{q}_i dp_i - dL \longrightarrow (3)$$

Again, $L = L(q_i, \dot{q}_i, t)$

$$\text{so, } dL = \frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt \longrightarrow (4).$$

Using (4) in (3) $\Rightarrow$

$$dH = p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt$$

$$\Rightarrow dH = \dot{q}_i dp_i - p_i dq_i - \frac{\partial L}{\partial t}dt \longrightarrow (5).$$

comparing equⁿ (5) with equⁿ (1)

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

This are known as Hamiltonian canonical equation

(Q) Sol: If a co-ordinate is cylic in Lagrangian mechanics it also cylic in Hamiltonian mechanics.

→ ~~$\frac{\partial q}{\partial}$~~ ~~$\frac{\partial L}{\partial q_i}$~~ ∂ canonical ϕ. equⁿ

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

if $\frac{\partial L}{\partial q_i} = 0$ then $\frac{\partial H}{\partial q_i} = 0$

(Q) if a co-ordinate q is cylic in Lagrangian mechanics then $\frac{\partial L}{\partial q} = 0 \Rightarrow \dot{p} = 0$

$$\dot{p}_i = -\frac{\partial H}{\partial q} \Rightarrow \frac{\partial H}{\partial q} = 0$$

$\therefore q$ is also cylic in Hamiltonian mechanics

(Q) Lagrangian of a system $L = \frac{1}{2} m(r^2\dot{\theta}^2 + \dot{r}^2) + \frac{K}{r}$ K = const. Find the Hamiltonian of a system.

→ p_r = Generalised momentum for generalised coordinate r, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\therefore p_r = m\dot{r}$

p_θ = Generalised momentum for generalised co-ordinate θ, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\therefore \dot{r} = \frac{p_r}{m}$, $\dot{\theta} = \frac{p_\theta}{mr^2}$

We know that Hamiltonian of a system, $H = \sum p_i \dot{q}_i - L$

$$= p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} m(r^2\dot{\theta}^2 + \dot{r}^2) - K/r$$

$$\Rightarrow H = \frac{p_\theta^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2} m\left(r^2 \frac{p_\theta^2}{m^2 r^4} + \frac{p_r^2}{m^2}\right) - \frac{K}{r}$$

$$= \frac{p_\theta^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2}\frac{p_\theta^2}{mr^2} - \frac{1}{2}\frac{p_r^2}{m} - \frac{K}{r}$$

$$H = \frac{1}{2}\frac{p_\theta^2}{m} + \frac{p_\theta^2}{2mr^2} - \frac{K}{r}$$

(Q). Lagrangian of a sys, $L = \frac{\dot{x}^2}{2x} + V(x)$, Find H.

$\longrightarrow$ p_x = generalised momentum for generalised co-ordinate. $p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{2} / \left(-\frac{1}{x^2}\right)$

$$= \frac{2\dot{x}}{2x} = \frac{\dot{x}}{x}$$

$$\dot{x} = p_x x \longrightarrow (1).$$

$$\therefore H = \sum \dot{x} p_x - L$$

$$= p_x \cdot \dot{x} - L$$

$$= (p_x x)^2 - L$$

$$= (p_x x)^2 - \frac{\dot{x}^2}{2x} + V(x)$$

$$= p_x^2 x^2 - \frac{p_x^2 x^2}{2x} + V(x)$$

$$= \frac{2p_x^2 x^3 - p_x^2 x^3}{2x} + V(x)$$

$$= p_x^2 x - \frac{p_x^2 x}{2} + V(x)$$

$$= \frac{1}{2} p_x^2 x + V(x)$$

H ✓

(Q). Hamiltonian of a sys $H = \sqrt{p^2 + m^2} + V(x)$
m = mass of particle. Find the Lagrangian func^n.

$\longrightarrow$ From canonical equ^n $\dot{q}_i = \frac{\partial H}{\partial p} = \dot{x}$

$$\therefore \frac{\partial H}{\partial p} = \frac{1 \times 2p}{2\sqrt{p^2 + m^2}} = \frac{p}{\sqrt{p^2 + m^2}} = \dot{x} \longrightarrow (1).$$

we know that $H = \sum p_i \dot{q}_i - L$

$$\Rightarrow L = \sum p_i \dot{q}_i - H$$

$$= p\dot{x} - \sqrt{p^2 + m^2} - V(x)$$

$$= \frac{\dot{x} m}{\sqrt{1 - \dot{x}^2}} - \sqrt{\frac{\dot{x}^2 m^2}{1 - \dot{x}^2} + m^2} - V(x)$$

$\dot{x} = \frac{p}{\sqrt{p^2 + m^2}}$

$\Rightarrow \dot{x}^2 = \frac{p^2}{p^2 + m^2}$

$\Rightarrow \frac{1}{\dot{x}^2} = \frac{p^2 + m^2}{p^2}$

$1 + \frac{m^2}{p^2} = \frac{1}{\dot{x}^2}$

$p^2 = \frac{\dot{x}^2 m^2}{1 - \dot{x}^2}$

$\Rightarrow \frac{m^2}{p^2} = \frac{1}{\dot{x}^2} - 1$

$\Rightarrow \frac{p^2}{m^2} = \frac{\dot{x}^2}{1 - \dot{x}^2}$

$$\Rightarrow L = \frac{\dot{x}^2 m}{\sqrt{1-\dot{x}^2}} - \sqrt{\frac{\dot{x}^2 m^2 + m^2 - m^2\dot{x}^2}{1-\dot{x}^2}} - V(x)$$

$$L = \frac{-m + m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - V(x)$$

$$\boxed{L = -m\sqrt{1-\dot{x}^2} - V(x)}$$

(Q). Given: $H = \frac{p^2}{(m-\lambda q)2}$ where m, λ are const. Find L.

$$\longrightarrow H = \frac{p^2}{2(m-\lambda q)}$$

$$H = \sum p_i \dot{q}_i - L$$

$$\Rightarrow H = p\dot{q} - L$$

$$\Rightarrow L = p\dot{q} - H$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{2p}{2(m-\lambda q)}$$

$$\dot{q} = \frac{p}{m-\lambda q}$$

$$\Rightarrow p = (m-\lambda q)\dot{q}$$

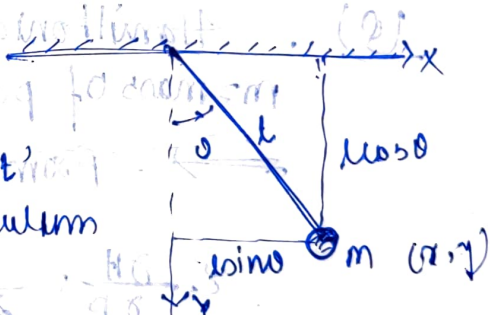
$$\therefore L = (m-\lambda q)\dot{q}^2 - \frac{(m-\lambda q)^2\dot{q}^2}{2(m-\lambda q)}$$

$$= (m-\lambda q)\dot{q}^2 - \frac{(m-\lambda q)}{2}\dot{q}^2$$

$$L = \frac{1}{2}(m-\lambda q)\dot{q}^2$$

$\vec{L} = \vec{r} \times \vec{p}$

(Q) Find Hamiltonian eqn of motion for simple pendulum.



$\longrightarrow$ We consider a simple pendulum of length l and mass $= m$ at any time 't'. The displacement of the pendulum is (x,y) from the origin.

From the fig, $x = l\sin\theta \Rightarrow \dot{x} = l\cos\theta\,\dot{\theta}$

$y = l\cos\theta \quad \dot{y} = -l\sin\theta\,\dot{\theta}$

$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m(l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2)$$

$$= \frac{m}{2}l^2\dot{\theta}^2 = \frac{ml^2\dot{\theta}^2}{2ml^2} = \frac{(ml^2\dot{\theta})^2}{2ml^2} = \frac{p_\theta^2}{2ml^2}$$

$$PE = mgy_1 = -mgl\cos\theta$$

$H = T + V$

$= \frac{P_\theta^2}{2mr^2} - mgl\cos\theta$

∴ Hamiltonian equⁿ of motion $\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2}$

$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$

H/W (Q) Find Hamilton equⁿ of motion for a particle in central force field.

⟹ $KE = \frac{1}{2}mv^2$ $\quad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\quad \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$PE = V(r)$

∴ $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

~~and Hamiltonian, $H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$ (1)~~

~~Again~~ $H = \sum_{i=1} P_i q_i - L$

$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$; $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\dot{r} = \frac{P_r}{m}$; $\dot{\theta} = \frac{P_\theta}{mr^2}$

∴ $H = P_r\dot{r} + P_\theta\dot{\theta} - L$

$H = \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \frac{m}{2}\frac{P_r^2}{m^2} - \frac{mr^2}{2}\cdot\frac{P_\theta^2}{m^2r^4} + V(r)$

$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$

Equation motion, $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

∴ $\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m}$; $\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2}$

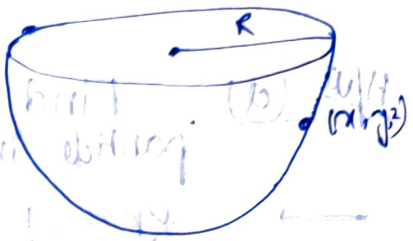
$\dot{P}_r = -\frac{\partial H}{\partial r} = \frac{P_\theta^2}{mr^3} + \frac{\partial V}{\partial r}$

$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = 0$

25-05

(9). Find the Lagrangian fun^n of a particle moving on the surface of a hemispherical shell of radius R with potential $V = \frac{1}{2}(x^2+y^2)$. Hence find the hamiltonian equ^n of motion.

→ let at any time t the pos^n of the particle is (x,y,z) and the mass of the particle is m



$x = R\sin\theta\cos\phi$
$y = R\sin\theta\sin\phi$
$z = R\cos\theta$

$0 \le \theta \le \pi/2$
$0 \le \phi < 2\pi$

$\therefore \dot{x} = R\sin\theta(-\sin\phi)\dot{\phi} + R\cos\phi\cos\theta\,\dot{\theta}$

$\dot{y} = R\sin\theta\cos\phi\,\dot{\phi} + R\sin\phi\sin\theta\,\dot{\theta}$

$\dot{z} = R(-\sin\theta)\dot{\theta}$

KE of the particle, $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$KE = \frac{m}{2}\left[(R\cos\phi\cos\theta\dot{\theta} - R\sin\theta\sin\phi\dot{\phi})^2 + (R\sin\theta\cos\phi\dot{\phi} + R\sin\phi\sin\theta\dot{\theta} + R^2\sin^2\theta\dot{\theta}^2\right]$

$= \frac{mR^2}{2}\left[\cos^2\phi\cos^2\theta\dot{\theta}^2 + \cos^2\theta\sin^2\phi\dot{\phi}^2 - 2\sin\theta\sin\phi\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\cos^2\phi\dot{\phi}^2 + \sin^2\phi\sin^2\theta\dot{\theta}^2 + 2\sin\theta\sin\phi\cos\theta\cos\phi\dot{\theta}\dot{\phi} + R^2\sin^2\theta\dot{\theta}^2\right]$

$= \frac{mR^2}{2}\left[\dot{\phi}^2\sin^2\theta(\sin^2\phi+\cos^2\phi) + \cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\theta}^2\right]$

$= \frac{mR^2}{2}\left[\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2\right]$

$T = \frac{1}{2}mR^2\sin^2\theta\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\theta}^2$

PE, $V = \frac{1}{2}(x^2+y^2)$

$= \frac{1}{2}\left\{R^2\sin^2\theta\cos^2\phi + R^2\sin^2\theta\sin^2\phi\right\} = \frac{R^2}{2}\sin^2\theta$

$L = T - V$

$L = \frac{1}{2} mR^2 \sin^2\theta \dot{\phi}^2 + \frac{1}{2} mR^2 \dot{\theta}^2 - \frac{R^2}{2} \sin^2\theta$

Here ϕ, θ are generalised co-ordinate

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mR^2}$

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2 \sin^2\theta \dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mR^2 \sin^2\theta}$

$\therefore$ Hamiltonian, $H = \sum p\dot{q} - L$

$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} mR^2 \sin^2\theta \dot{\phi}^2 - \frac{1}{2} mR^2 \dot{\theta}^2 + \frac{R^2}{2} \sin^2\theta$

$H = \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2 \sin^2\theta} - \frac{m}{2} R^2 \sin^2\theta \dot{\phi}^2 - \frac{mR^2 \dot{\theta}^2}{2} + \frac{R^2 \sin^2\theta}{2}$

$\therefore \dot{q} = \frac{\partial H}{\partial p_i}$ $\quad H = \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2 \sin^2\theta} - \frac{mR^2 \sin^2\theta}{2} \frac{p_\theta^2}{m^2 R^4 \sin^2\theta}$

$\Rightarrow \dot{\theta} = \frac{\partial H}{\partial p_\theta}$ $\quad - \frac{mR^2}{2} \frac{p_\theta^2}{m^2 R^4} + \frac{R^2}{2} \sin^2\theta$

$= \frac{2p_\theta}{mR^2}$ $\quad H = \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2 \sin^2\theta} - \frac{p_\theta^2 \sin^2\theta}{2mR^2 \sin^2\theta} -$

$\frac{p_\theta^2}{2mR^2} + R^2 \frac{\sin^2\theta}{2}$

$\Rightarrow H = \frac{1}{2} p_\theta^2 \frac{1}{mR^2} + \frac{1}{2} \frac{p_\phi^2}{mR^2 \sin^2\theta} + \frac{R^2 \sin^2\theta}{2}$

$\therefore \dot{\theta} = \frac{\partial H}{\partial p_\theta}$ $\quad \dot{\phi} = \frac{\partial H}{\partial p_\phi}$

$\dot{\theta} = \frac{p_\theta}{mR^2}$ $\quad \dot{\phi} = \frac{p_\phi}{mR^2 \sin^2\theta}$

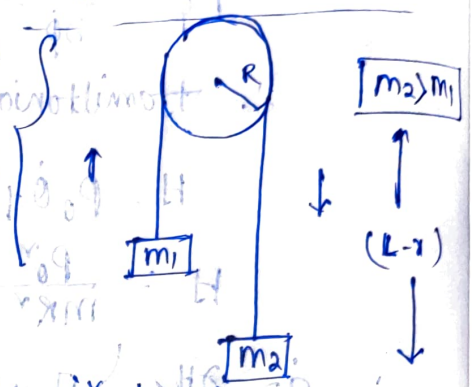
$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$

$= -R^2 \sin\theta\cos\theta - \frac{p_\phi^2}{2mR^2} (-2) \frac{\cos\theta}{\sin^3\theta}$

$\dot{p}_\theta = -R^2 \sin\theta\cos\theta + \frac{p_\phi^2 \cos\theta}{mR^2 \sin^3\theta}$

$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

(9). Two mass m_1 and m_2 is connected by a string of length L over a pulley of radius R. If the mass of the pulley is negligible find the Lagrangian equation of motion of the system. Hence find the acceleration of these two masses.



⟶ At any time t the distance of mass m_1 is x from reference

So the distance of mass m_2 from the reference is $(L-x)$

The KE of the system, $T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$

The PE of the system, $V = -m_1gx - m_2g(L-x)$

$$V = -m_1gx - m_2gL + m_2gx$$

$$\therefore L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_1gx + m_2gL - m_2gx$$

Here generalised co-ordinate is x

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}\{m_1\dot{x} + m_2\dot{x}\} - (m_1g - m_2g) = 0$$

$$\Rightarrow m_1\ddot{x} + m_2\ddot{x} + m_2g - m_1g = 0$$

$$\Rightarrow \ddot{x}(m_1+m_2) + g(m_2-m_1)$$

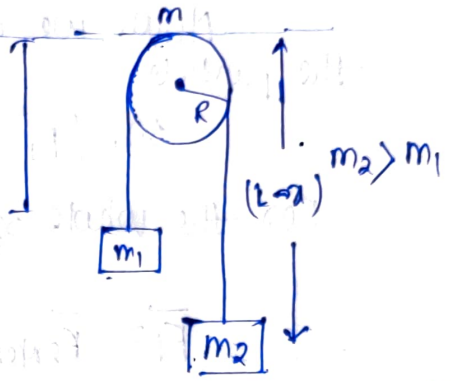
$$\therefore F = ma$$

$$\Rightarrow m_1g - m_2g = (m_1+m_2)a$$

$$\Rightarrow m_1g - m_2g = a(m_1+$$

$$\Rightarrow \ddot{x} = -\frac{g(m_2-m_1)}{m_1+m_2}$$

(9). Two mass m and 2m is connected by a string of length L over a pulley of radius R. If the mass of pulley is m, find the Lagrangian equn of motion of the sys. Hence find the acceleration of m and 2m



$\rightarrow$ KE, $T = \frac{m_1\dot{x}^2}{2} + \frac{m_2\dot{x}^2}{2} + \frac{1}{2}I\omega^2$

$= \frac{m_1\dot{x}^2}{2} + \frac{m_2\dot{x}^2}{2} + \cancel{\frac{1}{2}m_1R^2\omega^2} + \frac{1}{2}m_2R^2\omega^2$

$= \frac{m\dot{x}^2}{2} + \frac{2m\dot{x}^2}{2} + \frac{1}{2}mR^2\omega^2 \cdot \frac{1}{2}$

$T = \frac{3}{2}m\dot{x}^2 + \frac{1}{4}mR^2\omega^2$ $\quad I = \frac{1}{2}mR^2$

$= \frac{3}{2}m\dot{x}^2 + \frac{1}{4}m\dot{x}^2$ $\quad \omega = \frac{\dot{x}}{R}$

$T = \frac{7}{4}m\dot{x}^2$

$\therefore V = -mgx - 2mgL + 2mgx$

$= mgx - 2mgL$

$\therefore L = T - V$

$L = \frac{7}{4}m\dot{x}^2 - mgx + 2mgL$

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{7}{2}m\dot{x}\right) + mg = 0$

$\Rightarrow \frac{7}{2}m\ddot{x} + mg = 0$

$\Rightarrow \frac{7}{2}\ddot{x} + g = 0$

$\Rightarrow \ddot{x} = \frac{-2g}{7}$

(Q). Prove D'Alembert's principle.

$\rightarrow$ Consider the motion of an N particles system. Let the force on the i^{th} particle be F_i,

$F_i = \dot{P}_i \Rightarrow F_i - \dot{P}_i = 0 \longrightarrow (1)$.

equation (i) represent that the ith particle is at equilibrium if a 'reverse effective force' $\vec{p}_i$ is applied on the particle.

Under the action of "reverse effective force" a dynamic system is reduced to static system.

Now, we consider $\delta\vec{r}_i$ virtual displacement of the particle

So, $(\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$ ⟶ (ii).

For the whole system $\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$ ⟶ (iii).

$\vec{F}_i = \vec{F}_{external} + \vec{F}_{constraint}$ w.r.t $\vec{F}_{constraint} = 0$

$$\sum_{i=1}^{N} (\vec{F}_{ext} + \vec{F}_{constr} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\vec{F}_{ext} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$ ⟶ (iv). [work done by $F_{constr} = 0$]

eqn (iv) is known as D'Alembert's principle.

importance :- By using D'Alembert's principle we can convert a dynamic system to a static system.

H/W (q) Define configuration space and phase-space. write down the advantages of phase-space over configuration space.

⟶ configuration space :- The parameters that define the configuration of a system are called generalised co-ordinates and the space defined by these co-ordinates is called the configuration space of the physical system. i.e position of constituent particles.

phase space :- In a dynamical system, a phase space is a space in which all possible states of a system are represented with each possible state corresponding to one unique point in the phase space. For mechanical system

the ~~space~~ phase space usually consists of all possible value of position and momentum variables.

03-06